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Path-integral analysis of the propagator of two coupled graded-index waveguides

C C Constantinou† and Raymund C Jones‡

† School of Electronic and Electrical Engineering,
‡ School of Physics and Space Research, The University of Birmingham, Edgbaston, Birmingham B15 2TT, UK

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Abstract. The method of path integration is used to study coupled graded-index waveguides in the context of paraxial, scalar-wave optics. The cases of strong and intermediate strength coupling for waveguides of arbitrary and variable spacing are considered, and a variational estimate of the propagator of such a waveguide structure is derived in closed form. The complete variational calculation is presented for the case of parallel waveguides only, and the propagator, lowest-order-mode field profile, the first two lowest-order-mode propagation constants, and the beat length of the structure are determined in closed form.

1. Introduction

The propagation of optical waves in coupled waveguide systems is of great importance in the design of modern optical communication systems. Parallel and non-parallel waveguide couplers form an integral part of most optical systems, such as switches, power dividers, frequency and mode selectors and, therefore, filters and modulators. Since all communications systems essentially consist of switches, modulators, filters and transmission lines, it is evident that the performance of such structures critically affects the performance characteristics of optical communication networks in their entirety (Huang and Haus 1990). Detailed knowledge of the coupling between intersecting and branching waveguides in the region where they merge to form a taper (that is the region of strong coupling between the two waveguides), is well known to be very important in the design and operation of such structures (Burns and Milton 1975, 1990). Our ability to control the crosstalk (whether wanted or unwanted) between waveguides in close proximity determines, for example, whether an optical y-junction behaves as a power splitter or a mode selector/converter (Yajima 1974). Some examples of relevant waveguide structures such as switches and waveguide branches and intersections are shown schematically in figure 1.

Most of the work concerning the analysis of coupled waveguide systems tends to fall into one of three categories:

(i) Step-index waveguides (Marcatilli 1969, Marcuse 1982 – see section 10.3 in particular).
(ii) Weakly coupled waveguides, where the overlap integral between the modes of the individual (isolated) waveguides is used to find the coupling coefficient

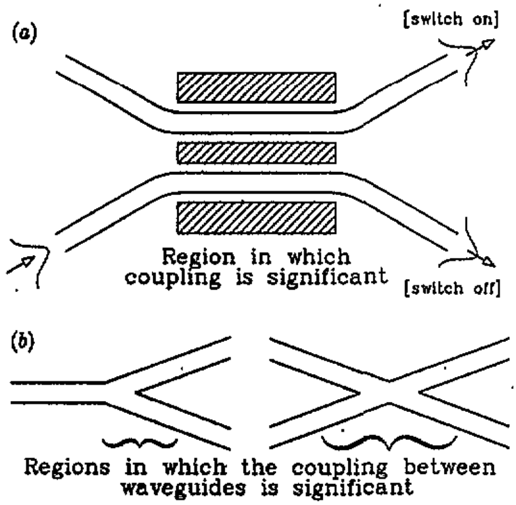

Figure 1. (a) A simple integrated optical switch: a directional coupler built on an electro-optic material has an effective length which varies according to the voltage applied on the electrodes (shown hatched in the drawing). (b) Schematic examples of waveguide junctions and branches: the regions indicated are the regions in which the two waveguides are either strongly coupled, or have merged into a single waveguide. These regions critically determine the propagation characteristics of each waveguide structure shown.

relevant to the problem (Miller 1954, Yariv 1973, Snyder and Love 1983, Hardy and Streifer 1985, Huang and Haus 1990, Burns and Milton 1990). The assumption of weak coupling is not always explicitly stated, nor is it always built into the formalism of the work to which we refer above, but is introduced implicitly by virtue of the fact that it is the modes of the uniform, individual, uncoupled waveguides which are used as basis functions in the analyses of these various problems. This is a point which has caused some controversy on the validity and limitations of 'conventional' coupled mode theory (Hardy and Streifer 1985, Haus *et al.* 1987).

(iii) Numerical simulations, such as the beam propagation method (Cullen 1985, Neyer *et al.* 1985).

In this paper we use the method of path integration (Feynman and Hibbs 1965) in conjunction with Feynman's variational technique in order to obtain in closed form the Green function, or propagator of the paraxial, scalar Helmholtz equation for two graded-index waveguides in close proximity. Path integration has been used as an analytical tool in the study of guided-wave optics over a number of years (Eichmann 1971, Eve 1976, Hannay 1977, Gómez-Reino and Liñares 1987, Constantinou 1991). A detailed discussion of the validity of the paraxial approximation and the appropriateness of its use in the modelling of integrated-optical waveguides, can be found in a series of papers, henceforth referred to as CJa, CJb, CJc and JC (Constantinou and Jones 1991a, b, 1992, Jones and Constantinou 1992). The model refractive index distribution we have chosen to work with is described in detail in the next section.

Although we begin our calculations by considering two graded-index waveguides whose separation can vary arbitrarily, we shall soon see that our results are expected to be valid only in the cases of strong and intermediate strength coupling and are thus complementary to much of the existing work. However, although most of our analysis concerns the case of variable waveguide separation, the maximization required by the variational technique employed here, can be performed rigorously only for parallel waveguides. An extension of the work to the case of non-parallel waveguides is proposed at the end of this paper.

This paper presents a number of new results. In section 5 we find a closed-form expression for the propagator of two parallel, strongly coupled graded-index waveguides. Expressions are presented for the lowest-order-mode field profile and for the propagation constants describing the fields of the two lowest-order modes of the two parallel waveguides. From these latter two propagation constants, we have also obtained a closed-form variational expression for the beat length of the two coupled waveguides. The beat length is the distance along the two parallel waveguides over which a complete energy exchange cycle takes place, and is therefore a quantity of considerable engineering interest (Snyder and Love 1983). Although the calculation presented here was carried out for a mathematically tractable idealization of the refractive index distribution corresponding to two coupled waveguides, the closed-form results arrived at for the propagator, beat length and lowest-order-mode field profile, provide us with insight into the strong coupling limit, by allowing us to see explicitly how the various quantities of interest depend on the refractive index parameters. To the best of our knowledge, strongly coupled waveguides have only been studied through numerical simulations (Cullen 1985). The present analysis is not intended to replace such simulation methods, which are in a sense more powerful since they can deal with arbitrary refractive index distributions, but is instead intended to complement them.

2. The model refractive index distribution

Two integrated optical graded-index waveguides which are in close proximity are usually manufactured by diffusing metallic atoms, such as silver or titanium, into a suitable substrate, such as $LiNbO_3$ or quartz. By taking a suitable cross-section of such an arrangement of waveguides, the refractive index distribution will have the general form shown in figure 2(a). Near the centres of each individual guide the refractive

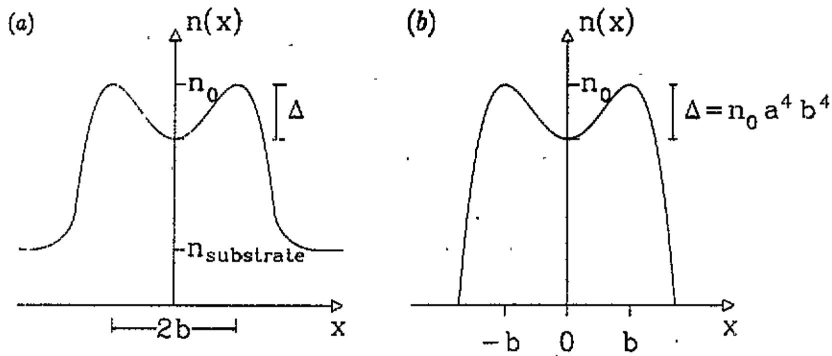

Figure 2. (a) The refractive index distribution of two parallel graded-index waveguides in close proximity, is depicted here schematically. In practice, Δ increases with the waveguide separation, $2b$, to a value $n_0 - n_{\text{substrate}}$. (b) The model refractive index distribution of equation (2.1).

index decreases roughly quadratically with distance from the centre of that guide. At large distances from either guide, the refractive index settles down to that of the substrate. It has been shown in CJa, CJb, CJc and JC that we can construct a mathematical model of a waveguide in which we allow the model refractive index distribution to take non-physical (negative) values at large distances from the guide, provided that *the fields themselves, and hence the energy transport, associated with these non-physical regions are very small, and* at the same time we restrict our considerations to multimode waveguides and waveguide modes of sufficiently low order. This assumption must be checked *a posteriori*, and is satisfied in the model calculations CJa, CJb, CJc and JC for a single guide. We construct a similar model to describe the problem of two parallel waveguides by modelling the physical distribution shown in figure 2(a) by a suitable chosen function (described below) which has the general form shown in figure 2(b). The above approximations make the problem analytically tractable. In CJa, CJb, CJc and JC (see also Gómez-Reino and Liñares 1987) we have demonstrated that, within the paraxial approximation, the full three-dimensional waveguide problem is separable and separates into two independent two-dimensional propagation problems. We define the z-axis of our chosen coordinate system to be parallel to the axes of the two waveguides, and the x-axis to be the axis passing through the centres of the two guides. Henceforth we will omit any y-dependent terms in the expressions for the refractive index and the propagator, and will instead concentrate on the xz-dependence of such functions only. Results for the full three-dimensional problem can be obtained by simply multiplying the appropriate results for the two independent *two-dimensional problems together* (CJa, CJb, CJc and JC).

The two-dimensional model refractive index distribution,

$$n(x) = n_0(1 - a^4(x^2 - b^2)^2) \tag{2.1}$$

which is plotted in Figure 2(b) has the correct shape required to give a reasonable description of the refractive index around and within the guides (Figure 2(a)). If we wish to consider coupled waveguides of arbitrary and varying separation $2b(z)$, we can re-write the above expression as,

$$n(x, z) = n_0(1 - a^4(x^2 - b^2(z))^2). \tag{2.2}$$

Equation (2.2) accurately models the situation in which, as the distance between the two waveguides increases, the refractive index between the waveguides, becomes ever smaller until the two waveguides are effectively isolated, in which case it reaches the value of $n_{\text{substrate}}$. In our model we effectively have a substrate for which $n_{\text{substrate}} \to -\infty$, which also guarantees waveguide isolation.

3. Theory

Paraxial propagation along the z-axis of a medium with refractive index distribution $n(x, z)$ is described by the Green function, or propagator, of the scalar Helmholtz equation which satisfies

$$\left(\frac{i}{k}\frac{\partial}{\partial z} + \frac{1}{2k^2}\frac{\partial^2}{\partial x^2} + \frac{n(x, z)}{n_0}\right) K(x, z; x_0, z_0) = \delta(x - x_0)\delta(z - z_0) \tag{3.1}$$

It has been shown by a number of authors (Eichmann, 1971, Eve, 1976, Schulman

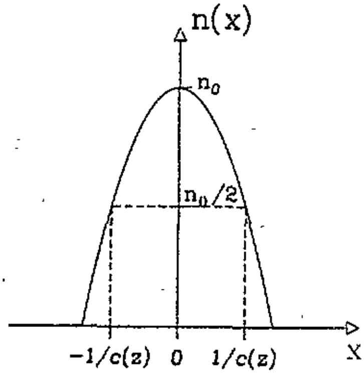

Figure 3. The refractive index distribution of equation (3.3) used in the variational calculation.

1981, Gómez-Reino and Liñares 1987, CJa, CJb, CJc, JC), that the propagator can be written as the path integral

$$K(x,z;x_0,z_0)=\exp[ik(z-z_0)]\int_{x(z_0)=x_0}^{x(z)=x}\delta x(z)\exp\left\{ik\int_{z_0}^{z}\mathrm{d}\zeta\left(\frac{\dot{x}^2(\zeta)}{2}-a^4[x^2(\zeta)-b^2(\zeta)]^2\right)\right\} \tag{3.2}$$

in which

$$\int_{x(z_0)=x_0}^{x(z)=x}\delta x(z)$$

represents a sum over all optical ray paths which begin at the point (x_0, z_0) and pass through the point (x, z) (Feynman and Hibbs 1965). The wavenumber k is defined to be $k \equiv 2\pi n_0/\lambda_0$, where λ_0 is the free-space wavelength. Unfortunately it is not possible to evaluate the above path integral in closed form exactly, because the exponent in (3.2) is quartic in the path variable $x(\zeta)$. An approximate evaluation of the above path integral is possible though, if we make use of the Feynman variational technique (Feynman and Hibbs 1965), which we briefly summarize below.

A variational approximation to the propagator of a waveguide having a refractive index distribution described by equation (2.2), can be derived by considering the propagator of a simpler waveguide structure, for which we can evaluate exactly the corresponding path integral. The simpler (and soluble) waveguide structure is chosen so that it shares a number of features of optical propagation with the model refractive index distribution (2.2) which are felt to be important. We choose this model refractive index used in such a variational calculation to be

$$n(x,z)=n_0\left(1-\frac{1}{2}c^2(z)x^2\right). \tag{3.3}$$

It is well known that light tends to concentrate in regions of locally high refractive index (Born and Wolf 1980). The refractive index distribution (3.3) (shown in figure 3) tends to concentrate the light around its maximum on the z-axis. When the two waveguides are sufficiently close (i.e. when $2b(z)$ is small and the coupling between the guides is strong), the model refractive index distribution (2.2) tends to concentrate the light around the two maxima which are close to the z-axis. Thus both (2.2) and (3.3) share the property of confining light in the vicinity of the z-axis of the chosen coordinate system. Both refractive index distributions then have broadly similar waveguiding properties. The spacing between the two waveguides in (2.2) is $2b(z)$, the width of the 'variational waveguide' in (3.3) is proportional to $1/c(z)$: both are arbitrary functions of the z-coordinate—an important common feature of the two refractive index distributions. There is no *a priori* relation between $b(z)$ and $c(z)$. The above two common features make the refractive index distribution (3.3) suitable for use in the Feynman variational technique in order to determine an approximate expression for the propagator of the refractive index distribution (2.2), and is likely to be a sensible approximation when the guides are strongly coupled.

In order to state the variational principle for this problem, we must first define a number of quantities: the simpler and soluble refractive index distribution used in this technique (i.e. the refractive index distribution (3.3) in our case), will be called the trial refractive index, and all the quantities associated with it (e.g. its associated trial propagator), will be labelled with the subscript t. The optical path length associated with a refractive index distribution, $n(x, z)$, is defined in the usual way to be

$$S=\int_{z_0}^{z} d\zeta \left(\frac{\dot{x}^2(\zeta)}{2}+n(x(\zeta),\zeta)\right). \tag{3.4}$$

We will also need the functional average, denoted by angular brackets $\langle\cdot\rangle$ and defined by

$$\langle\mathcal{F}[x(z)]\rangle=\frac{\displaystyle\int \delta x(z)\mathcal{F}[x(z)]\exp\{ikS_t\}}{\displaystyle\int \delta x(z)\exp\{ikS_t\}}. \tag{3.5}$$

Feynman's variational technique (Feynman and Hibbs 1965) as modified by Samathiyakanit (Samathiyakanit 1972, Constantinou 1991) states that the quantity

$$K(x,z;x_0,z_0)\simeq K_t(x,z;x_0,z_0)\exp[ik\langle S-S_t\rangle] \tag{3.6}$$

can be used as a satisfactory approximation to the propagator of the medium with refractive index $n(x)$, provided that we maximize the lowest-order-mode propagation constant, β_0, of this latter medium

$$\beta_0\simeq\beta_{t0}+\lim_{\mu\to-\infty}\frac{k}{\mu}\langle S-S_t\rangle \tag{3.7}$$

with respect to any free variational parameter; μ is the analytically continued value of $-i(z-z_0)$ and β_{t0} is the lowest-order-mode propagation constant of the simpler variational waveguide. This is analogous to the process of minimizing the ground state energy level in variational calculations in quantum mechanics, the only difference being that the spectrum of modal propagation constants is bounded from above rather than below as is the case in mechanics (Constantinou 1991). It is evident that this formulation of the variational technique requires that all the refractive index distributions be independent of the variable z, in order to ensure that the quantities β_0 and β_{t0}

be well defined quantities. The problem of refractive index distributions which vary with z is not rigorously resolved in this paper—an approximate solution for adiabatically varying waveguides is suggested in section 6 instead.

4. The calculation of the approximate propagator

Without loss of generality we neglect a pre-factor $\exp[ik(z-z_0)]$ which occurs in the full propagator expression and arises from the non-zero, constant part, n_0, of the refractive index; the propagator corresponding to the trial refractive index distribution (3.3) is then given by

$$K_t(x,z;x_0,z_0)=\int \delta x(z)\exp\left\{ik\int_{z_0}^{z} d\zeta\left[\frac{\dot{x}^2(\zeta)}{2}-\frac{c^2(\zeta)x^2(\zeta)}{2}\right]\right\}. \qquad (4.1)$$

We will hereafter omit all such exponential pre-factors from all subsequent propagator expressions for the sake of brevity. The above path integral can be readily evaluated (cf, Gómez-Reino and Liñares 1987) and gives,

$$K_t(x,z;x_0,z_0)=\left(\frac{k}{2\pi i f}\right)^{1/2}\exp\left\{ik/2\left(\frac{\partial \ln f}{\partial z}x^2-\frac{\partial \ln f}{\partial z_0}x_0^2-\frac{2xx_0}{f}\right)\right\} \qquad (4.2)$$

where $f\equiv f(z,z_0)$ satisfies the differential equation,

$$\frac{\partial^2 f(z,z_0)}{\partial z^2}+c^2(z)f(z,z_0)=0 \qquad (4.3)$$

with the boundary conditions

$$f(z=z_0,z_0)=0 \qquad \text{and} \qquad \left.\frac{\partial f(z,z_0)}{\partial z}\right|_{z=z_0}=1. \qquad (4.4)$$

Since the term

$$\int_{z_0}^{z} d\zeta \frac{\dot{x}^2(\zeta)}{2}$$

which is common to both the optical path lengths S and S_t, cancels out in equations (3.6) and (3.7), it is convenient to redefine the optical path lengths S and S_t, so that they are now given by

$$S=-a^4\int_{z_0}^{z} d\zeta[x^2(\zeta)-b^2(\zeta)]^2 \qquad (4.5)$$

$$S_t=-\frac{1}{2}\int_{z_0}^{z} d\zeta\, c^2(\zeta)x^2(\zeta). \qquad (4.6)$$

in order to simplify the calculation of the factor $\exp[ik\langle S-S_t\rangle]$ in equation (3.6). Furthermore, the average $\langle\cdot\rangle$ defined in (3.5) is now given by

$$\langle\mathcal{F}[x(z)]\rangle\equiv\frac{\int \delta x(z)\mathcal{F}[x(z)]\exp\left\{ik\int_{z_0}^{z} d\zeta\left(\frac{\dot{x}^2(\zeta)}{2}-\frac{c^2(\zeta)x^2(\zeta)}{2}\right)\right\}}{\int \delta x(z)\exp\left\{ik\int_{z_0}^{z} d\zeta\left(\frac{\dot{x}^2(\zeta)}{2}-\frac{c^2(\zeta)x^2(\zeta)}{2}\right)\right\}}. \qquad (4.7)$$

The denominator of the above expression is identical to (4.1) and its closed-form expression is therefore given by (4.2). The averaged quantities $\langle S\rangle$ and $\langle S_t\rangle$ can be expressed solely in terms of $\langle x^2(\zeta)\rangle$ and $\langle x^4(\zeta)\rangle$. The latter two averages are readily computed from the characteristic functional Φ of the process $x(\zeta)$ (Feynman and Hibbs 1965), defined by

$$\Phi \equiv \left\langle \exp\left\{ ik \int_{z_0}^{z} \mathrm{d}\zeta\, g(\zeta) x(\zeta) \right\} \right\rangle \tag{4.8}$$

where $g(\zeta)$ is an arbitrary, continuous function of ζ. Successive functional differentiations of Φ with respect to $g(\zeta)$, show that,

$$\langle x^n(\zeta)\rangle = \frac{1}{(ik)^n} \frac{\delta^n \phi}{\delta g(\zeta)^n}\bigg|_{g(\zeta)=0}. \tag{4.9}$$

In appendix 1 we present the detailed evaluation of expression (4.8), which yields

$$\Phi = \exp\left\{ \frac{ikx}{f(z, z_0)} \int_{z_0}^{z} \mathrm{d}\zeta g(\zeta) f(\zeta, z_0) + \frac{ikx_0}{f(z, z_0)} \int_{z_0}^{z} \mathrm{d}\zeta g(\zeta) f(z, \zeta) \right.$$
$$\left. + \frac{1}{2} \int_{z_0}^{z} \mathrm{d}\zeta \int_{z_0}^{z} \mathrm{d}\zeta' g(\zeta) g(\zeta') G(\zeta; \zeta') \right\} \tag{4.10}$$

where $G(\zeta; \zeta')$ is the Green function of equation (4.3) with homogeneous boundary conditions at $\zeta = z_0$ and $\zeta = z$, and is defined formally and derived in closed form in appendix 1. Using equation (4.9) and functionally differentiating (4.10) with respect to $g(\zeta)$, then gives the following closed-form expressions for $\langle x^2(\zeta)\rangle$ and $\langle x^4(\zeta)\rangle$:

$$\langle x^2(\zeta)\rangle = \left(\frac{xf(\zeta, z_0) + x_0 f(z, \zeta)}{f(z, z_0)} \right)^2 - \frac{f(z, \zeta) f(\zeta, z_0)}{ikf(z, z_0)} \tag{4.11}$$

and

$$\langle x^4(\zeta)\rangle = \left(\frac{xf(\zeta, z_0) + x_0 f(z, \zeta)}{f(z, z_0)} \right)^4 - 6 \frac{f(z, \zeta) f(\zeta, z_0)}{ikf(z, z_0)} \left(\frac{xf(\zeta, z_0) + x_0 f(z, \zeta)}{f(z, z_0)} \right)^2$$
$$+ 3 \left(\frac{f(z, \zeta) f(\zeta, z_0)}{ikf(z, z_0)} \right)^2. \tag{4.12}$$

Finally, we see from (4.5) and (4.6) that the difference between the optical path lengths associated with the trial and exact propagators for the coupled waveguide system is given by

$$ik\langle S - S_t\rangle = -ika^4 \int_{z_0}^{z} \mathrm{d}\zeta b^4(\zeta) + ik \int_{z_0}^{z} \mathrm{d}\zeta\, [2a^4 b^2(\zeta) + c^2(\zeta)/2] \langle x^2(\zeta)\rangle$$
$$- ika^4 \int_{z_0}^{z} \mathrm{d}\zeta \langle x^4(\zeta)\rangle. \tag{4.13}$$

Substituting for the terms $\langle x^2(\zeta)\rangle$ and $\langle x^4(\zeta)\rangle$ from equations (4.11) and (4.12), shows that equation (4.13) can be written as,

$$
\begin{aligned}
ik\langle S-S_t\rangle = & -ika^4\int_{z_0}^{z} \mathrm{d}\zeta\, b^4(\zeta) - \frac{ika^4}{f^4(z,z_0)}\int_{z_0}^{z} \mathrm{d}\zeta[xf(\zeta,z_0)+x_0 f(z,\zeta)]^4 \\
& -\frac{3a^4}{ikf^2(z,z_0)}\int_{z_0}^{z} \mathrm{d}\zeta f^2(z,\zeta)f^2(\zeta,z_0) \\
& +\frac{6a^4}{f^3(z,z_0)}\int_{z_0}^{z} \mathrm{d}\zeta f(z,\zeta)f(\zeta,z_0)[xf(\zeta,z_0)+x_0 f(z,\zeta)]^2 \\
& +\frac{ik}{f^2(z,z_0)}\int_{z_0}^{z} \mathrm{d}\zeta[2a^4b^2(\zeta)+c^2(\zeta)/2][xf(\zeta,z_0)+x_0 f(z,\zeta)]^2 \\
& -\frac{1}{f(z,z_0)}\int_{z_0}^{z} \mathrm{d}\zeta[2a^4b^2(\zeta)+c^2(\zeta)/2]f(z,\zeta)f(\zeta,z_0).
\end{aligned} \tag{4.14}
$$

An approximate final closed-form expression for the coupled waveguide propagator is then obtained by combining equations (3.6), (4.2) and (4.14), and is,

$$
\begin{aligned}
K(x,z;x_0,z_0) \simeq & \left(\frac{k}{2\pi i f(z,z_0)}\right)^{1/2} \exp\left\{ik\left((z-z_0)-a^4\int_{z_0}^{z}\mathrm{d}\zeta\, b^4(\zeta)\right.\right. \\
& \left.\left. +\frac{3a^4}{k^2f^2(z,z_0)}\int_{z_0}^{z}\mathrm{d}\zeta f^2(z,\zeta)f^2(\zeta,z_0)\right)\right\} \\
& \times\exp\left\{-\frac{1}{f(z,z_0)}\int_{z_0}^{z}\mathrm{d}\zeta[2a^4b^2(\zeta)+c^2(\zeta)/2]f(z,\zeta)f(\zeta,z_0)\right\} \\
& \times\exp\left\{ik/2\left(x^2\frac{\partial\ln f(z,z_0)}{\partial z}-x_0^2\frac{\partial\ln f(z,z_0)}{\partial z_0}-\frac{2xx_0}{f(z,z_0)}\right)\right\} \\
& \times\exp\left\{\frac{6a^4}{f^3(z,z_0)}\int_{z_0}^{z}\mathrm{d}\zeta f(z,\zeta)f(\zeta,z_0)[xf(\zeta,z_0)+x_0f(z,\zeta)]^2\right\} \\
& \times\exp\left\{\frac{ik}{f^2(z,z_0)}\left(\int_{z_0}^{z}\mathrm{d}\zeta[2a^4b^2(\zeta)+c^2(\zeta)/2][xf(\zeta,z_0)+x_0f(z,\zeta)]^2\right.\right. \\
& \left.\left.-\frac{a^4}{f^2(z,z_0)}\int_{z_0}^{z}\mathrm{d}\zeta[xf(\zeta,z_0)+x_0f(z,\zeta)]^4\right)\right\}.
\end{aligned} \tag{4.15}
$$

To the best of our knowledge, the above approximate but closed-form expression for the propagator of a model of two coupled graded-index waveguides is entirely new. Using the well known analogy between optics and mechanics (Constantinou 1991) which establishes a correspondence between the refractive index in optics and the potential in mechanics, we see that in the context of quantum mechanics, expression (4.15) gives an approximate form for the propagator of an anharmonic oscillator with strong coupling between the individual wells of the double well. It is well known (Schulman 1981) that the description of the motion of an anharmonic oscillator is closely linked to problems such as instantons in quantum field theory and second-order phase transitions in statistical mechanics. Therefore, the above new result has potential uses outside the context in which we now present it.

The propagator (4.15) does not constitute the best variational approximation to the propagation problem in the coupled graded-index waveguides system, until the optimal value of the function $c(z)$ and thus of $f(z, z_0)$ is used in (4.15). As we have already seen in section 3, the Feynman variational technique requires that we maximize the quantity β_0 calculated in the limit $(z-z_0)\to -i\infty$. The lowest-order-mode propagation constant, β_0, and the trial medium lowest-order-mode propagation constant, β_{t0}, are quantities which can only be formally defined for waveguides of uniform cross-section (Snyder and Love 1983). The case in which both β_0 and β_{t0} can be formally defined corresponds to two parallel graded-index waveguides. In this relatively simple problem we replace the functions $b(z)$ and $c(z)$ in equations (2.2) and (3.3) by the two scalar parameters b and c, respectively, which are independent of z. We then determine the lowest-order propagation constant β_0 in terms of these parameters, and finally maximize β_0 with respect to c. The more interesting case where $b(z)$ and $c(z)$ are arbitrary functions of z cannot be treated exactly here. In a separate section at the end of this paper we propose an ansatz for completing the variational calculation and speculate on possible ways to complete the variational calculation more formally.

An important property of the propagator (4.15) is that it contains a number of exponential terms, some of which have real exponents, and some of which have imaginary exponents. The solutions of equation (4.3) for *real* functions $c(z)$ are always oscillatory in nature, since the function $f(z, z_0)$ and its second derivative with respect to the variable z are always of opposite sign. The presence of oscillatory terms in the real exponents implies that at any given transverse coordinate position x, the amplitude of the propagating wave will alternately increase and then decrease with increasing z. This is precisely what we expect to happen in waveguides which are in close proximity: their fields are coupled and as a consequence, there is energy exchange between them (Snyder and Love 1983). As we will shortly see, when the waveguides are parallel the exchange is exactly periodic in z.

5. The approximate propagator and modes of two parallel, coupled graded-index waveguides

When we are considering two parallel, coupled graded-index waveguides, their separation $2b(z)$ is independent of z. We may therefore set $b(z)\equiv b$, and $c(z)\equiv c$, where both b and c are now constants. In this case the taper function $f(z, z_0)$ defined by (4.3) and (4.4) is simply given by,

$$f(z, z_0)=\frac{1}{c}\sin(c(z-z_0)). \tag{5.1}$$

The integrals of $f(z, z_0)$ which appear in the expression for the coupled waveguide propagator (4.15) are simple trigonometric integrals which can be readily evaluated to give

$$
\begin{aligned}
K(x,z;x_0,z_0) \simeq {} & \left(\frac{kc}{2\pi i\sin(c(z-z_0))}\right)^{1/2} \exp\left\{-\left(\frac{a^4b^2}{c^2}+\frac{1}{4}\right)(1-c(z-z_0)\cot(c(z-z_0)))\right\} \\
& \times \exp\left\{ik(1-a^4b^4)(z-z_0)\right. \\
& +\frac{9ia^4}{8kc^3\sin(c(z-z_0))}\left(\frac{c(z-z_0)}{\sin(c(z-z_0))}\left[1-\frac{2}{3}\sin^2(c(z-z_0))\right]\right. \\
& \left.\left.-\cos(c(z-z_0))\right)\right\} \\
& \times \exp\left\{\frac{9a^4}{4c^2\sin^2(c(z-z_0))}\left((x^2+x_0^2)\left(\left[1-\frac{1}{3}\sin^2(c(z-z_0))\right]\right.\right.\right. \\
& \left.-c(z-z_0)\cot(c(z-z_0))\right) \\
& \left.\left.+2xx_0\left(\frac{c(z-z_0)}{\sin(c(z-z_0))}\left[1-\frac{2}{3}\sin^2(c(z-z_0))\right]-\cos(c(z-z_0))\right)\right)\right\} \\
& \times \exp\left\{\frac{ikc}{\sin(c(z-z_0))}\left((x^2+x_0^2)\left(\frac{1}{2}\cos(c(z-z_0))\right.\right.\right. \\
& \left.+\left(\frac{a^4b^2}{c^2}+\frac{1}{4}\right)\left(\frac{c(z-z_0)}{\sin(c(z-z_0))}-\cos(c(z-z_0))\right)\right) \\
& \left.\left.+2xx_0\left(-\frac{1}{2}+\left(\frac{a^4b^2}{c^2}+\frac{1}{4}\right)(1-c(z-z_0)\cot(c(z-z_0)))\right)\right)\right\} \\
& \times \exp\left\{-\frac{ik}{\sin^3(c(z-z_0))}\left(\frac{3a^4}{8c}\right)\left((x^4+x_0^4)\left(\frac{c(z-z_0)}{\sin(c(z-z_0))}\right.\right.\right. \\
& \left.-\cos(c(z-z_0))\left[1+\frac{2}{3}\sin^2(c(z-z_0))\right]\right) \\
& +4xx_0(x^2+x_0^2)\left(\left[1-\frac{1}{3}\sin^2(c(z-z_0))\right]-c(z-z_0)\cot(c(z-z_0))\right) \\
& \left.\left.+6x^2x_0^2\left(\frac{c(z-z_0)}{\sin(c(z-z_0))}\left[1-\frac{2}{3}\sin^2(c(z-z_0))\right]-\cos(c(z-z_0))\right)\right)\right\}. \quad (5.2)
\end{aligned}
$$

This closed-form result giving an approximate propagator of two coupled graded-index waveguides is, to the best of our knowledge, new. Although the approximate propagator of the anharmonic oscillator in quantum mechanics has been derived in the past using other methods (Schulman 1981, Wiegel 1986), Feynman's variational method has never been applied to this problem before.

The propagator (5.2) exhibits the important feature to which we alluded briefly at the end of the previous section. With the exception of a transient response for small $(z-z_0)$, all the exponential terms in (5.2) are periodic in $(z-z_0)$. This periodic repeat distance is known in optical engineering as the beat length, z_b (Snyder and Love 1983), and is given by

$$z_b \equiv 2\pi/c. \tag{5.3}$$

The simple dependence of the beat length on the 'variational waveguide' parameter c is a result which could not be anticipated *a priori* and can only be arrived at through the use of the Feynman variational technique and the detailed examination of the resulting propagator expression (5.2). The functional dependence of z_b on the parameters a and b will be fixed when we complete the variational maximisation of β_0 which is defined in (3.7). The final result is given in (5.14). It is important to stress that this process of maximizing β_0 does not in any way imply that either the refractive index profiles or the modal filed distributions of the exact and variational guides are matched. The beat length is an important quantity which we must be able to predict accurately in order to design useful devices such as directional couplers (Lee 1986, Snyder and Love 1983, Tamir 1990). Inaccurate determination of the beat length can result in designing power dividers, modulators and switches with undesirable extinction ratios, or even worse improper operating characteristics resulting in unwanted crosstalk in an optical communication system.

We are now in a position to perform the maximization required by the variational method in order to obtain c and through (5.3), the beat length z_b. In order to maximize the lowest-order-mode propagation constant of the coupled waveguide structure, we first need to make the analytic continuation

$$z-z_0=i\mu \tag{5.4}$$

and consider the limit of large negative μ. In this limit, we have,

$$\sin(c(z-z_0))=\sin(i\mu c)\simeq\frac{\exp(-\mu c)}{2i}, \tag{5.5}$$

and

$$\cos(c(z-z_0))=\cos(i\mu c)\simeq\frac{\exp(-\mu c)}{2}. \tag{5.6}$$

The expression (5.2) for the propagator then becomes,

$$\begin{aligned}K(x,z;x_0,z_0)\simeq\left(\frac{2ikc}{2\pi i\exp(-c\mu)}\right)^{1/2}\exp\Bigg(&-k\mu+ka^4b^4\mu+\frac{3a^4\mu}{4kc^2}-\left(\frac{a^4b^2}{c^2}+\frac{1}{4}\right)\mu c\\&+\frac{9a^4}{8kc^3}-\left(\frac{a^4b^2}{c^2}+\frac{1}{4}\right)-\left(\frac{a^4b^2}{c^2}-\frac{1}{4}\right)kc(x^2+x_0^2)\\&-\frac{3a^4}{4c^2}(x^2+x_0^2)-\frac{ka^4}{c}(x^4+x_0^4)\Bigg)\end{aligned} \tag{5.7}$$

where we have negelected all the terms multiplied by any power of $\exp(+\mu c)$. Taking the natural logarithm of the above expression, dividing by $-\mu$ and letting $\mu \to -\infty$ finally gives

$$\beta_0 \simeq \lim_{\mu \to -\infty} \left(-\frac{1}{\mu} \ln(K(x, z; x_0, z_0)) \right) = (k - c/2) + \left(\frac{a^4 b^2}{c^2} + \frac{1}{4} \right) c - k a^4 b^4 - \frac{3a^4}{4kc^2}. \tag{5.8}$$

The first two terms $k - c/2$ constitute the lowest-order-mode propagation constant, β_{t0}, of the variational trial waveguide (see section 3). According to the variational principle, we must now maximize β_0 with respect to c. Thus, we need to solve the equation

$$\frac{d\beta_0}{dc} = -\frac{1}{2} - \frac{a^4 b^2}{c^2} + \frac{1}{4} + \frac{3a^4}{2kc^3} = 0 \tag{5.9}$$

for the parameter c. In order to ensure that the value of c given by (5.9) makes β_0 a maximum, it must also satisfy,

$$\frac{d^2\beta_0}{dc^2} = \frac{2a^4 b^2}{c^3} - \frac{9a^4}{2kc^4} < 0 \tag{5.10}$$

which implies that

$$c < \frac{9}{4kb^2}. \tag{5.11}$$

Thus in order to determine c we must solve the cubic equation,

$$c^3 + 2a^4 b^2 c - 3a^4/k = 0 \tag{5.12}$$

for one of its real roots, which must be less than $c < 9/4kb^2$. The discriminant of the cubic equation D can be easily determined (Abramowitz and Stegun 1965, paragraph 3.8.2) and is found to be given by

$$D = \frac{8a^{12} b^6}{27} + \frac{9a^8}{4k^2} \tag{5.13}$$

which is always positive. This implies that the cubic equation has only one real root given by (Abramowitz and Stegun 1965, paragraph 3.8.2),

$$c = \left(\frac{3a^4}{2k} \right)^{1/3} \left(\left(\left(1 + \left(\frac{2}{3} \right)^5 k^2 a^4 b^6 \right)^{1/2} + 1 \right)^{1/3} - \left(\left(1 + \left(\frac{2}{3} \right)^5 k^2 a^4 b^6 \right)^{1/2} - 1 \right)^{1/3} \right). \tag{5.14}$$

A change of variable to $t \equiv ((\frac{2}{3})^5 k^2 a^4 b^6)^{1/6}$, transforms equation (5.14) and the inequality (5.11) to

$$c = \left(\frac{3a^4}{2k} \right)^{1/3} (((1 + t^6)^{1/2} + 1)^{1/3} - ((1 + t^6)^{1/2} - 1)^{1/3}) \tag{5.15}$$

and

$$c < \left(\frac{3a^4}{2k} \right)^{1/3} t^{-2} \tag{5.16}$$

respectively. The inequality (5.16) thus becomes,

$$((1 + t^6)^{1/2} + 1)^{1/3} - ((1 + t^6)^{1/2} - 1)^{1/3} < t^{-2}. \tag{5.17}$$

This inequality holds for all values of t in the range $0 \leqslant t < +\infty$, as can be seen by expanding the LHS of (5.17) into an infinite series into the variable $1/t$. Thus we see that the value of c in (5.14) is always the optimal solution to the variational problem, for all values of the parameters a, b and k.

The explicit form of the dimensionless parameter t is worth examining here, because it gives us some insight into the physical parameters governing the coupling mechanism. t is proportional to $(kb/\pi)^{1/3}$, where kb/π is the separation of the two guides measured in wavelengths, and to $(a^4b^4)^{1/6}$, where a^4b^4 is the ratio of the depth of the refractive index on the z-axis to its peak value at the centre of the two waveguides. The fractional depth in the refractive index on the z-axis corresponds to the height of the potential barrier in the quantum mechanical problem of electronic motion in a double potential well. The two dimensionless parameters a^4b^4 and kb/π are also known from other work (Landau and Lifshitz 1977, Wiegel 1973) to be important in determining c. The qualitative dependence of c on these parameters predicted by all methods of analysis (including ours) is that the beat length increases monotonically with the separation of the two guides and the fractional depth of the refractive index between them.

The expression for the parameter c (5.14) has a number of important features worth discussing. For the sake of convenience in the discussion below, we define the corresponding dimensionless parameter c' by $c' \equiv (k/3a^4)^{1/3}c$. We can easily see that

$$c' = \left(\frac{(1+t^6)^{1/2}+1}{2}\right)^{1/3} - \left(\frac{(1+t^6)^{1/2}-1}{2}\right)^{1/3}. \tag{5.18}$$

A plot of c' against t is shown in figure 4. We see that for any given fixed values of a and λ, c' is a monotonically decreasing function of b. For the particular choice of $a = n_0/\lambda_0\sqrt{2}$ (an extreme case) and $b = \lambda_0/n_0$, it is found that $1/c \approx 5\lambda_0/n_0$. A nonlinear least-squares fit algorithm shows that the optimum description of the above curve is

$$c' \simeq \exp\{-0.7t^{1.58}\} \tag{5.19}$$

which resembles neither an exponential, nor a Gaussian function. In figure 4 we have also plotted, for the sake of comparison, the curve described by equation (5.19) as well as the exponential and Gaussian curves which best fit the exact solution.

To the best of our knowledge there exists only one path-integral analysis of motion in a double potential well, and we believe this latter analysis to be cruder than our variational calculation. This latter approximate method was developed by Wiegel (1973, 1975) in his study of Brownian motion in a field of force, and is called the hopping paths approximation. Briefly, the hopping paths approximation consists of the following logical steps: the Brownian particle (corresponding to a ray of light in the optical problem) spends most of its time at the bottom of the adjacent potential wells and thus the classical action corresponding to this section of its path can be calculated easily. We then assume that the particle 'hops' between the bottoms of these two adjacent potential wells at discrete times t_1, t_2, t_3, etc. The hopping paths approximation results in c' being described by an exponential function, which does not agree with our result (5.14).

Rather more conventional approximate analyses, such as the weak coupling approximation, using differential equations, also tend to give a result which is an exponential function of some kind (Landau and Lifshitz 1977, Marcuse 1982, Lee 1986, Burns and Milton 1990).

Using equation (5.18) we can see that for large values of the dimensionless parameter t, $c' \approx (2/3t^2)$. The physical significance of having a large value of t is that it corresponds either to well separated waveguides, or to waveguides separated by a very deep region of low refractive index. Therefore, the limit of large t is that of the weak coupling approximation. Our results predict that the beat length increases as t^2, whereas most other analyses predict at least an exponential rise for large t. This discrepancy arises from the fact that for large separations and/or well isolated waveguides, the parabolic refractive index distribution which we have used as the starting point in the variational calculation ceases to be an acceptable trial variational approximation to the refractive index profile shown in figure 2(b). Therefore, in the limit $t \gg 1$ our result is not as reliable as those resulting from other analyses (Wiegel 1973, 1975, Landau and Lifshitz 1977, Marcuse 1982, Lee 1986, Burns and Milton 1990). Nevertheless, in the limit of small t, or equivalently the case of strongly coupled waveguides, our model is likely to be more reliable than the models described above, since its derivation does not involve any simplifying assumptions. It should be stressed that the limit $t \ll 1$, or equivalently $(k^2a^4b^6)^{1/6} \ll 1$, is to be treated as a criterion and not as a quantitative estimate for the validity of the strong coupling theory. Unfortunately, we have not found any experimental data which would check whether our predictions are better or worse than those of existing theories. In principle an experiment along the lines of the one described in Feit *et al.* (1983) together with the method of Schmidt and Kaminow (1974) for measuring diffusant concentrations can be devised to measure the beat length of strongly coupled waveguides and their refractive index distributions, respectively, and hence test our theoretical predictions. It would, of course, be necesary to determine values of the parameters a and b in equation (2.1) in order to obtain a best fit between the measured refractive index distribution and the one described by equation (2.1). Alternatively, the predictions of our approach could be compared to numerical simulations. We are presently looking into comparisons with numerical simulations such as the beam propagation method (Neyer *et al.* 1985).

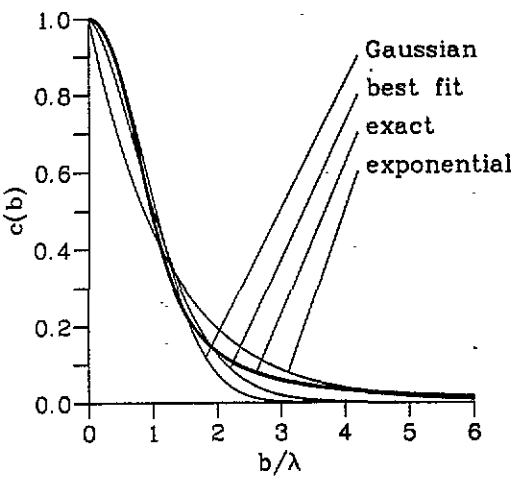

Figure 4. The dimensionless coupling parameter c' for the two coupled graded-index waveguide problem, plotted against the guide separation parameter t. The exact curve corresponding to equation (5.18) is shown together with the various fitted exponential-type curves for comparison.

It is well known (see for example Feynman and Hibbs 1965) that useful information, such as the various modal field profiles and their propagation constants, can be extracted from the closed-form expression for the propagator of any waveguiding structure whose cross-sectional refractive index distribution is invariant along the z-axis. Furthermore, the closed-form expression for the propagator can be used to study the propagation of any scalar field distribution through such waveguiding structures. In this paper we limit our considerations to the study of the two lowest-order modes of the two parallel, coupled graded-index waveguides only.

The procedure we use for extracting the information about the modes of the coupled waveguide system is that presented in chapter 8 of Feynman and Hibbs (1965). Expanding all the trigonometric functions in the expression for the propagator (5.2) into their Maclaurin series in the variable $\exp(-ic(z-z_0))$, and retaining only terms which are at most of first order in this variable, we have,

$$\begin{aligned}K(x,z;x_0,z_0)\simeq&\left(\frac{kc}{\pi}\right)^{1/2}\exp\left\{-\frac{a^4b^2}{c^2}-\frac{1}{4}+\frac{9a^4}{8kc^3}\right\}\\&\times\exp\left\{ikz-icz/2-ika^4b^4z+i\left(\frac{a^4b^2}{c^2}+\frac{1}{4}\right)cz-\frac{3ia^4}{4kc^2}z\right\}\\&\times\exp\left\{-\frac{3a^4}{4c^2}(x^2+x_0^2)-\frac{kc}{2}(x^2+x_0^2)+\left(\frac{a^4b^2}{c^2}+\frac{1}{4}\right)kc(x^2+x_0^2)\right.\\&\left.-\frac{3a^4k}{4c}(x^4+x_0^4)\right\}\\&\times\exp\left\{\left(-\frac{6a^4}{c^2}icz+4ikc^2z\left(\frac{a^4b^2}{c^2}+\frac{1}{4}\right)+\frac{9a^4}{c^2}+2kc-4kc\left(\frac{a^4b^2}{c^2}+\frac{1}{4}\right)\right.\right.\\&\left.\left.-\frac{a^4k}{c}(x^2+x_0^2)\right)e^{-icz}xx_0\right\}\end{aligned}\tag{5.20}$$

where we have set $z_0=0$, without loss of generality. Expanding the last exponential term into its infinite power series, rearranging the z-dependent terms and then resumming, yields

$$\begin{aligned}K(x,z;x_0,z_0)\simeq&\left(\frac{kc}{\pi}\right)^{1/2}\exp\left\{-\frac{a^4b^2}{c^2}-\frac{1}{4}+\frac{9a^4}{8kc^3}\right\}\\&\times\exp\left\{ikz-icz/2-ika^4b^4z+i\left(\frac{a^4b^2}{c^2}+\frac{1}{4}\right)cz-\frac{3ia^4}{4kc^2}z\right\}\\&\times\exp\left\{-\frac{3a^4}{4c^2}(x^2+x_0^2)-\frac{kc}{2}(x^2+x_0^2)+\left(\frac{a^4b^2}{c^2}+\frac{1}{4}\right)kc(x^2+x_0^2)\right.\\&\left.-\frac{3a^4k}{4c}(x^4+x_0^4)\right\}\\&\times\left[1+xx_0\left(\frac{9a^4}{c^2}-4kc\left(\frac{a^4b^2}{c^2}-\frac{1}{4}\right)-\frac{a^4k}{c}(x^2+x_0^2)\right)e^{-icz}+O(e^{-2icz})\right].\end{aligned}\tag{5.21}$$

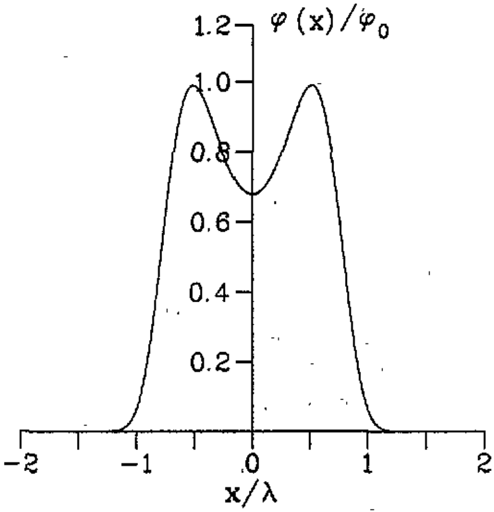

Figure 5. The lowest-order-mode field profile for two coupled graded-index waveguides for which $b=\lambda_0/n_0$ and $a=n_0/(\lambda_0\sqrt{2})$.

Comparing the above result with the standard eigenfunction expansion of the propagator (Feynman and Hibbs 1965)

$$K(x,z;x_0,z_0)=\sum_{n=0}^{\infty}\varphi_n(x)\varphi_n^*(x_0)\exp[i\beta_n(z-z_0)] \tag{5.22}$$

where $\varphi(x)$ corresponds to any one of the Cartesian components of the electric, magnetic field vector, the magnetic vector potential, or the scalar potential, we can see that the lowest-order-mode propagation constant and field profile predicted by our method are given by

$$\beta_0\simeq k-c/4-ka^4b^4+a^4b^2/c-\frac{3a^4}{4kc^2} \tag{5.23}$$

$$\varphi_0(x)\simeq\left(\frac{kc}{\pi}\right)^{1/4}\exp\left\{-\frac{a^4b^2}{2c^2}-\frac{1}{8}+\frac{9a^4}{16kc^3}\right\}\exp\left\{\left(\left(\frac{a^4b^2}{c^2}+\frac{1}{4}\right)kc-\frac{3a^4}{4c^2}\right)x^2-\left(\frac{3a^4k}{4c}\right)x^4\right\} \tag{5.24}$$

respectively. A typical plot of $\varphi_0(x)$ against x is shown in figure 5. It is worth pointing out that the two peaks in the field distribution occur at $x\simeq\pm\lambda/2$, while the corresponding peaks in the refractive index distribution occur at $x=\pm\lambda$. This shifting of the position of the peaks of the field amplitude towards each other is a consequence of the strong coupling between the two waveguides. Most conventional analyses of coupled waveguides (e.g. Snyder and Love 1983) consider the unperturbed fields of each guide in isolation and estimate the coupling parameter c by finding an overlap integral between the modes of the two waveguides. Clearly the presence of this significant shifting of the field maxima makes the implicit assumption involved in the conventional coupled mode analyses invalid: we cannot define in any meaningful way the modes of a single waveguide in the presence of a second waveguide in close proximity. The propagation constant of the first excited odd mode is finally given by,

$$\beta_1 \simeq k - 5c/4 - ka^4b^4 + a^4b^2/c - \frac{3a^4}{4kc^2}. \tag{5.25}$$

Because of the fact that the variational technique optimizes the fit between the lowest-order-mode propagation constants of the exact and trial refractive index distributions, the modal field distributions for higher-order modes which we can extract are necessarily much cruder approximations to the true eigenfunctions. This shortcoming manifests itself even more strongly in the case under study, since here we cannot even write down an expression for the field profile of the first excited mode. This is due to the presence of the term $(a^4k/c)(x^2+x_0^2)$ in the expression for $\varphi_1(x)\varphi_1^*(x_0)$, (cf equation (5.21)), which is not separable in the variables x and x_0. The expression for the propagation constant of this mode (5.25) provided by the variational technique is however expected to be an accurate upper bound, since the product $\varphi_1(x)\varphi_1^*(x_0)$ is orthogonal to the corresponding lowest-order-mode product $\varphi_0(x)\varphi_0^*(x_0)$ for both the exact and approximate eigenfunctions $\varphi_0(x)$ and $\varphi_1(x)$ (Sakurai 1985). In spite of this failure of the variational method, the presence of a common factor xx_0 in (5.21) enables us to predict that the first excited mode of the coupled waveguide system must have a node at $x=0$.

Since the lowest-order mode is an even mode and the first excited mode is an odd mode, their sum and difference turn out to represent wave distributions which are localized in the waveguides centred at the points $x=+b$ and $x=-b$, respectively. The propagation constant difference $\Delta\beta=\beta_0-\beta_1$ can be seen from expressions (5.23) and (5.25) to be given by $\Delta\beta=c$, which confirms that the propagator expression (5.2) predicts the periodic exchange of energy between the two coupled waveguides, with a beat length equal to $2\pi/c$.

6. Two non-parallel, coupled graded-index waveguides: speculations on a possible way forward

We have so far seen that Feynman's variational method requires that in order to obtain the optimum parameter c for a given waveguide separation distance $2b$, we must define a propagation constant β_0 which we maximize with respect to c. It is not possible to generalize this method to the case when c and b are functions of the paraxial propagation distance z, because in this case β_0 is not a quantity which can be defined in any sensible way (Snyder and Love 1983). Furthermore, even if it were possible to define some quantity which should be maximized with respect to $c(z)$, the resulting calculation would involve calculating a functional derivative with respect to $c(z)$, while at the same time taking into account the functional dependence of $f(z, z_0)$ on $c(z)$. We are currently exploring ways of doing this, and we will not consider it any further in this paper.

One way forward is to make the conjecture that we can match the exact and trial parabolic refractive index distributions in each and every transverse plane to the z-axis, and so write

$$c(a,b(z),k)=c(z)$$
$$=\left(\frac{3a^4}{2k}\right)^{1/3}\left(\left(\left(1+\left(\frac{2}{3}\right)^5 k^2a^4b^6(z)\right)^{1/2}+1\right)^{1/3}\right.$$
$$\left.-\left(\left(1+\left(\frac{2}{3}\right)^5 k^2a^4b^6(z)\right)^{1/2}-1\right)^{1/3}\right). \tag{6.1}$$

It should be clear at this stage that substituting expression (6.1) into the propagator (5.2) will give nonsense, unless the parameter $b(z)$ varies sufficiently slowly with z so that the expression

$$\frac{\mathrm{d}[c(z)z]}{\mathrm{d}z}\approx c(z) \tag{6.2}$$

holds approximately for all values of z in the range of interest. Another way of expressing the criterion (6.2) is to write it in the form $\mathrm{d}b(z)/\mathrm{d}z \ll 1$. When the above criterion is satisfied, the waveguide system under study is undergoing what Burns and Milton (1990) have described as an adiabatic waveguide transition, and in this case the concept of local normal modes becomes applicable.

Obviously, the adiabatic approximation involves making two distinct assumptions: First we make a conjecture (6.1) which determines the function $c(z)$, and then we substitute this result into the expression for the propagator of two *parallel* waveguides (5.2). Our approach allows us to make an improved adiabatic approximation in the sense that after determining the function $c(z)$ in (6.1), we can substitute it into the differential equation (4.3) and find $f(z,z_0)$. Explicit knowledge of the function $f(z,z_0)$, then enables us to determine the full expression (4.15) for the propagator of the system of two coupled waveguides with a variable spacing. The differential equation for $f(z,z_0)$ which we need to solve, is then

$$\frac{\partial^2 f(z,z_0)}{\partial z^2}+\left[\left(\frac{3a^4}{2k}\right)^{1/3}\left(\left(\left(1+\left(\frac{2}{3}\right)^5 k^2a^4b^6(z)\right)^{1/2}+1\right)^{1/3}\right.\right.$$
$$\left.\left.-\left(\left(1+\left(\frac{2}{3}\right)^5 k^2a^4b^6(z)\right)^{1/2}-1\right)^{1/3}\right)\right]^2 f(z,z_0)=0. \tag{6.3}$$

Its solutions must satisfy the boundary conditions (4.4). Although it may be difficult to find closed-form solutions of the differential equation (6.3) for a number of separation functions $b(z)$, the asymptotic expansion of $c(z)$ in the limit of small t or even the use of the WKB approximation may result in closed-form solutions for $f(z,z_0)$. This calculation is clearly very lengthy and is at present in progress. We hope to report on this in a future publication.

7. Conclusions

In this paper we have presented a refractive index model for a coupled, graded-index waveguide system in which the spacing between the two waveguides is variable. This model is described mathematically by equation (2.2) and is plotted in figure 2(b). The most important feature which we have tried to build into this model is that the region between the two waveguides should have a refractive index which decreases rapidly when the separation of the two waveguides increases.

We have applied the path-integral formalism to the coupled waveguide system in conjunction with the Feynman variational technique in order to obtain an approximate closed form for its propagator. The trial refractive index distribution which we used in the variational technique was that of an infinite parabolic refractive index tapered waveguide of arbitrary geometry. The closed-form expression for the variationally obtained approximate propagator of the coupled waveguide system with an arbitrary spacing is, to the best of our knowledge, entirely new.

The special case of the propagator of the system of two parallel coupled waveguides was then considered in some detail, and new results were obtained for the beat length of the two waveguides, together with information on the propagation constants and physically sensible mode field profiles of the two lowest-order modes of this structure. On theoretical grounds, we suggest that for strong and intermediate strengths of the coupling, our results are likely to predict a better approximation for the beat length compared with that produced by other analyses which largely are relevant to weak coupling problems.

Finally, we have identified the shortcomings of Feynman's variational technique in determining the propagator of two coupled waveguides of varying separation, and we have proposed an ansatz for determining the propagator of two waveguides which separate adiabatically. Work on this is currently in progress.

Work is in hand on the comparison of our predictions with the predictions of standard numerical methods such as the beam propagation method applied to our model refractive index distribution.

Appendix 1. The determination of the characteristic functional Φ

Using the definition of the average $\langle\cdot\rangle$ given in equation (4.7) and the definition of Φ in (4.8), we can see that the numerator of the expression for Φ is given by,

$$I=\int \delta x(z)\exp\left\{ik\int_{z_0}^{z} d\zeta\left(\frac{\dot{x}^2(\zeta)}{2}-\frac{c^2(\zeta)x^2(\zeta)}{2}+g(\zeta)x(\zeta)\right)\right\}. \qquad \text{(A1.1)}$$

The above path integral is the propagator of a forced quantum mechanical harmonic oscillator for which the external force g and the spring stiffness c are both arbitrary functions of time, which in this case corresponds to the spatial coordinate variable ζ. To the best of our knowledge this quantum mechanical problem has never been solved in the past, possibly because it does not apply to any physical problem of interest in mainstream physics. The propagator (A1.1) only differs from that in (4.1) by the presence of a term in the exponent which is linear in $x(\zeta)$. Using the arguments given by Feynman and Hibbs (1965) and Schulman (1981) for evaluating quadratic functionals, the path integral in (A1.1) can be readily evaluated to give,

$$I=\left(\frac{k}{2\pi if}\right)^{1/2}\exp\{ikS_{\mathrm{GO}}\} \qquad \text{(A1.2)}$$

where f is defined in (4.3) and S_{GO} is the optical path length of the ray path $X(\zeta)$ prescribed by geometrical optics:

$$S_{GO}=\int_{z_0}^{z} d\zeta\left(\frac{\dot{X}^2(\zeta)}{2}-\frac{c^2(\zeta)X^2(\zeta)}{2}+g(\zeta)X(\zeta)\right). \tag{A1.3}$$

Using Fermat's principle (Born and Wolf 1980), we can immediately see that the geometrical optics ray path $X(\zeta)$ is the solution of the Euler–Lagrange equation (Jeffreys and Jeffreys 1956)

$$\frac{d^2X(\zeta)}{d\zeta^2}+c^2(\zeta)X(\zeta)=g(\zeta) \tag{A1.4}$$

and satisfies the boundary conditions

$$X(z_0)=x_0 \qquad \text{and} \qquad X(z)=x. \tag{A1.5}$$

The closed-form solution for $X(\zeta)$ can be found by writing it as

$$X(\zeta)=X_1(\zeta)+X_2(\zeta) \tag{A1.6}$$

where $X_1(\zeta)$ satisfies the homogeneous differential equation (A1.4) with the inhomogeneous boundary conditions (A1.5), and $X_2(\zeta)$ satisfies the inhomogeneous differential equation (A1.4) with homogeneous boundary conditions. By virtue of the fact that the function $f(z, z_0)$ satisfies the same differential equation (4.3), and the boundary conditions (4.4), we may express $X_1(\zeta)$ in terms of $f(z, z_0)$, as

$$X_1(\zeta)=\frac{xf(\zeta, z_0)+x_0 f(z, \zeta)}{f(z, z_0)}. \tag{A1.7}$$

$X_2(\zeta)$ can be easily determined using the Green function, $G(\zeta; \zeta')$, defined by

$$\left(\frac{d^2}{d\zeta^2}+c^2(\zeta)\right)G(\zeta; \zeta')=\delta(\zeta-\zeta'). \tag{A1.8}$$

This Green function can also be expressed in terms of $f(z, z_0)$. It is a straightforward matter to show that

$$G(\zeta; \zeta')=\begin{cases} -\dfrac{f(\zeta, z_0)f(z, \zeta')}{f(z, z_0)} & \text{for } z_0<\zeta<\zeta' \\ -\dfrac{f(\zeta', z_0)f(z, \zeta)}{f(z, z_0)} & \text{for } \zeta'<\zeta<z. \end{cases} \tag{A1.9}$$

The function $X_2(\zeta)$ is then given by

$$X_2(\zeta)=\int_{z_0}^{z} d\zeta'\, g(\zeta')\, G(\zeta; \zeta'). \tag{A1.10}$$

Combining the results (A1.7), (A1.9) and (A1.10), we obtain the following expression for the geometrical optics ray path $X(\zeta)$,

$$X(\zeta)=\frac{1}{f(z,z_0)}\left(xf(\zeta,z_0)+x_0f(z,\zeta)-f(\zeta,z_0)\int_\zeta^z d\zeta' g(\zeta')f(z,\zeta')\right.$$
$$\left.-f(z,\zeta)\int_{z_0}^\zeta d\zeta' g(\zeta')f(\zeta',z_0)\right) \qquad \text{(A1.11)}$$

Substituting equation (A1.11) into (A1.3), then determines S_{GO}:

$$S_{GO}=\frac{1}{2f(z,z_0)}\left\{\frac{1}{f(z,z_0)}\left[x^2\int_{z_0}^z d\zeta\left[\left(\frac{\partial f(\zeta,z_0)}{\partial\zeta}\right)^2-c^2(\zeta)f^2(\zeta,z_0)\right]\right.\right.$$
$$+x_0^2\int_{z_0}^z d\zeta\left[\left(\frac{\partial f(z,\zeta)}{\partial\zeta}\right)^2-c^2(\zeta)f^2(z,\zeta)\right]$$
$$+2xx_0\int_{z_0}^z d\zeta\left[\frac{\partial f(\zeta,z_0)}{\partial\zeta}\frac{\partial f(z,\zeta)}{\partial\zeta}-c^2(\zeta)f(\zeta,z_0)f(z,\zeta)\right]$$
$$-2x\int_{z_0}^z d\zeta\left\{\left[\left(\frac{\partial f(\zeta,z_0)}{\partial\zeta}\right)^2-c^2(\zeta)f^2(\zeta,z_0)\right]\left[\int_\zeta^z d\zeta' g(\zeta')f(z,\zeta')\right]\right\}$$
$$-2x_0\int_{z_0}^z d\zeta\left\{\left[\left(\frac{\partial f(z,\zeta)}{\partial\zeta}\right)^2-c^2(\zeta)f^2(z,\zeta)\right]\left[\int_{z_0}^\zeta d\zeta' g(\zeta')f(\zeta',z_0)\right]\right\}$$
$$-2x\int_{z_0}^z d\zeta\left\{\left[\frac{\partial f(\zeta,z_0)}{\partial\zeta}\frac{\partial f(z,\zeta)}{\partial\zeta}\right.\right.$$
$$\left.\left.-c^2(\zeta)f(\zeta,z_0)f(z,\zeta)\right]\left[\int_{z_0}^\zeta d\zeta' g(\zeta')f(\zeta',z_0)\right]\right\}$$
$$-2x_0\int_{z_0}^z d\zeta\left\{\left(\frac{\partial f(\zeta,z_0)}{\partial\zeta}\frac{\partial f(z,\zeta)}{\partial\zeta}\right.\right.$$
$$\left.\left.-c^2(\zeta)f(\zeta,z_0)f(z,\zeta)\right]\left[\int_\zeta^z d\zeta' g(\zeta')f(z,\zeta')\right]\right\}$$
$$+\int_{z_0}^z d\zeta\left\{\left[\left(\frac{\partial f(\zeta,z_0)}{\partial\zeta}\right)^2-c^2(\zeta)f^2(\zeta,z_0)\right]\left[\int_\zeta^z d\zeta' g(\zeta')f(z,\zeta')\right]^2\right\}$$
$$+\int_{z_0}^z d\zeta\left\{\left[\left(\frac{\partial f(z,\zeta)}{\partial\zeta}\right)^2-c^2(\zeta)f^2(z,\zeta)\right]\left[\int_{z_0}^\zeta d\zeta' g(\zeta')f(\zeta' m\, z_0)\right]^2\right\}$$
$$+2\int_{z_0}^z d\zeta\left\{\left[\frac{\partial f(\zeta,z_0)}{\partial\zeta}\frac{\partial f(z,\zeta)}{\partial\zeta}\right.\right.$$
$$\left.\left.\left.-c^2(\zeta)f(\zeta,z_0)f(z,\zeta)\right]\left[\int_\zeta^z d\zeta' g(\zeta')f(z,\zeta')\right]\left[\int_{z_0}^\zeta d\zeta' g(\zeta')f(\zeta',z_0)\right]\right\}\right]$$
$$+2x\int_{z_0}^z d\zeta g(\zeta)f(\zeta,z_0)+2x_0\int_{z_0}^z d\zeta g(\zeta)f(z,\zeta)$$

$$-2\int_{z_0}^{z} d\zeta g(\zeta)f(\zeta, z_0)\int_{\zeta}^{z} d\zeta' g(\zeta')f(z, \zeta')$$

$$-2\int_{z_0}^{z} d\zeta g(\zeta)f(z, \zeta)\int_{z_0}^{\zeta} d\zeta' g(\zeta')f(\zeta', z_0)\bigg\}. \tag{A1.12}$$

All the integrals containing the term $-c^2(\zeta)$ in the expression for S_{GO} above can be evaluated by parts. Since it would be too tedious and lengthy to reproduce such calculations even in an appendix, we demonstrate the detailed evaluation of only one term, in order to illustrate the method used. All of the above integrals can be performed using the same general approach. Let us consider the evaluation of the integral, J, defined below, and which appears in the sixth and seventh lines of equation (A1.12), i.e.

$$J \equiv \int_{z_0}^{z} d\zeta\left\{\left[\frac{\partial f(\zeta, z_0)}{\partial \zeta}\frac{\partial f(z, \zeta)}{\partial \zeta} - c^2(\zeta)f(\zeta, z_0)f(z, \zeta)\right]\left[\int_{z_0}^{\zeta} d\zeta' g(\zeta')f(\zeta', z_0)\right]\right\}. \tag{A1.13}$$

The defining differential equation (4.3) for the function $f(z, z_0)$ can be used to substitute for $-c^2(\zeta)f(\zeta, z_0)$ in the above expression, to give

$$J = \int_{z_0}^{z} d\zeta\left\{\frac{\partial f(\zeta, z_0)}{\partial \zeta}\frac{\partial f(z, \zeta)}{\partial \zeta}\int_{z_0}^{\zeta} d\zeta' g(\zeta')f(\zeta', z_0)\right.$$

$$\left. + \frac{\partial^2 f(\zeta, z_0)}{\partial \zeta^2}f(z, \zeta)\int_{z_0}^{\zeta} d\zeta' g(\zeta')f(\zeta', z_0)\right\}. \tag{A1.14}$$

Integrating the second term in the integrand in (A1.14) by parts, finally yields

$$J = \int_{z_0}^{z} d\zeta\left\{\frac{\partial f(\zeta, z_0)}{\partial \zeta}\frac{\partial f(z, \zeta)}{\partial \zeta}\int_{z_0}^{\zeta} d\zeta' g(\zeta')f(\zeta', z_0)\right.$$

$$-\frac{\partial f(\zeta, z_0)}{\partial \zeta}\left(\frac{\partial f(z, \zeta)}{\partial \zeta}\int_{z_0}^{\zeta} d\zeta' g(\zeta')f(\zeta', z_0)\right.$$

$$\left.\left. + f(z, \zeta)g(\zeta)f(\zeta, z_0)\right)\right\} + \left[\frac{\partial f(\zeta, z_0)}{\partial \zeta}f(z, \zeta)\int_{z_0}^{\zeta} d\zeta' g(\zeta')f(\zeta', z_0)\right]_{\zeta = z_0}^{\zeta = z}. \tag{A1.15}$$

Using the boundary conditions (4.4) for $f(z, z_0)$, the above expression then simplifies to,

$$J = \int_{z_0}^{z} d\zeta g(\zeta)f(z, \zeta)f(\zeta, z_0)\frac{\partial f(\zeta, z_0)}{\partial \zeta}. \tag{A1.16}$$

The same general approach can be used to evaluate all the integrals in (A1.12). If we now use the expression for the Wronskian of $f(z, z_0)$ (appendix 2), we can group some of the resulting terms together to finally obtain

$$S_{GO}=\frac{1}{2}\left(x^2\frac{\partial \ln f(z,z_0)}{\partial z}-x_0^2\frac{\partial \ln f(z,z_0)}{\partial z_0}-\frac{2xx_0}{f(z,z_0)}\right)+\frac{1}{f(z,z_0)}\left(x\int_{z_0}^{z}d\zeta g(\zeta)f(\zeta,z_0)\right.$$
$$+x_0\int_{z_0}^{z}d\zeta g(\zeta)f(z,\zeta)-\frac{1}{2}\int_{z_0}^{z}d\zeta\int_{\zeta}^{z}d\zeta' g(\zeta)g(\zeta')f(z,\zeta')f(\zeta,z_0)$$
$$\left.-\frac{1}{2}\int_{z_0}^{z}d\zeta\int_{z_0}^{\zeta}d\zeta' g(\zeta)g(\zeta')f(z,\zeta)f(\zeta',z_0)\right). \qquad \text{(A1.17)}$$

The above expression for the optical path length can be further simplified if we use the explicit form (A1.9) for the Green function (A1.8), to get

$$S_{GO}=\frac{1}{2}x^2\frac{\partial \ln f(z,z_0)}{\partial z}-\frac{1}{2}x_0^2\frac{\partial \ln f(z,z_0)}{\partial z_0}-\frac{xx_0}{f(z,z_0)}$$
$$+\frac{x}{f(z,z_0)}\int_{z_0}^{z}d\zeta g(\zeta)f(\zeta,z_0)+\frac{x_0}{f(z,z_0)}\int_{z_0}^{z}d\zeta g(\zeta)f(z,\zeta)$$
$$+\frac{1}{2}\int_{z_0}^{z}d\zeta\int_{z_0}^{z}d\zeta' g(\zeta)g(\zeta')G(\zeta;\zeta'). \qquad \text{(A1.18)}$$

Using equations (4.3), (A1.2) and (A1.18), we arrive at the following expression for the characteristic functional Φ:

$$\Phi=\exp\left\{\frac{ikx}{f(z,z_0)}\int_{z_0}^{z}d\zeta g(\zeta)f(\zeta,z_0)+\frac{ikx_0}{f(z,z_0)}\int_{z_0}^{z}d\zeta g(\zeta)f(z,\zeta)\right.$$
$$\left.+\frac{1}{2}\int_{z_0}^{z}d\zeta\int_{z_0}^{z}d\zeta' g(\zeta)g(\zeta')G(\zeta;\zeta')\right\}. \qquad \text{(A1.19)}$$

Appendix 2. Some properties of the function $f(z, z_0)$

In CJc we have shown that the function $f(z, z_0)$ obeys the differential equation (4.3) together with the boundary conditions (4.4), which we rewrite below:

$$\frac{\partial^2 f(z,z_0)}{\partial z^2}+c^2(z)f(z,z_0)=0 \qquad \text{(A2.1)}$$

$$f(z=z_0,z_0)=0 \qquad \text{and} \qquad \left.\frac{\partial f(z,z_0)}{\partial z}\right|_{z=z_0}=1. \qquad \text{(A2.2)}$$

We denote by $\Xi_1(\zeta)$ and $\Xi_2(\zeta)$ the two linearly independent solutions of the second-order ordinary differential equation

$$\frac{d^2\Xi(\zeta)}{d\zeta^2}+c^2(\zeta)\Xi(\zeta)=0. \qquad \text{(A2.3)}$$

Fitting the boundary conditions (A2.2) results in,

$$f(z,z_0)=\frac{\Xi_1(z)\Xi_2(z_0)-\Xi_1(z_0)\Xi_2(z)}{\Xi_1'(z_0)\Xi_2(z_0)-\Xi_1(z_0)\Xi_2'(z_0)} \qquad \text{(A2.4)}$$

where a prime represents a differentiation of the appropriate function with respect to its argument. It should be noted that the denominator in the expression (A2.4) for $f(z, z_0)$ is the negative of the Wronskian of $\Xi_1(\zeta)$ and $\Xi_2(\zeta)$. By virtue of the fact that the differential equation (A2.1) has no first derivative term in it, the Wronskian

$$W\{\Xi_1(\zeta),\ \Xi_2(\zeta)\} \equiv \frac{d\Xi_1(\zeta)}{d\zeta}\Xi_2(\zeta) - \Xi_1(\zeta)\frac{d\Xi_2(\zeta)}{d\zeta} \tag{A2.5}$$

is independent of ζ (Morse and Feshbach 1953). Making use of this fact it is easy to show that

$$W\{f(z, \zeta), f(\zeta, z_0)\} = f(z, z_0). \tag{A2.6}$$

A very important symmetry property of $f(z, z_0)$ which follows directly from (A2.4) is,

$$f(z, z_0) = -f(z_0, z). \tag{A2.7}$$

Furthermore, if we define

$$F(z, z_0) \equiv \frac{\partial f(z, z_0)}{\partial z} \tag{A2.8}$$

equation (A2.4) further implies that

$$\frac{\partial f(z, z_0)}{\partial z_0} = -F(z_0, z). \tag{A2.9}$$

Finally, the function $f(z, z_0)$ also obeys the differential equation

$$\frac{\partial^2 f(z, z_0)}{\partial z_0^2} + c^2(z_0) f(z, z_0) = 0 \tag{A2.10}$$

with boundary conditions

$$f(z, z_0 = z) = 0 \qquad \text{and} \qquad \left.\frac{\partial f(z, z_0)}{\partial z_0}\right|_{z_0 = z} = -1. \tag{A2.11}$$

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